SOCIAL CHOICE AND COHERENT STRUCTURES

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ABSTRACT

The purpose of this paper is to show the relevance of reliability theory to the problem of aggregating individual preferences to social preferences over a set of alternatives. First, the Arrow impossibility theorem is proved using coherent structure arguments. Second, coherent systems as decision structures are examined and their properties studied. It is shown that only the self-dual systems are never inconsistent or blocked. It is further found that for any given coherent decision structure any set of alternatives is split into four subsets that have certain interesting properties. Finally, the introduction of probabilities indicates how a cardinal ordering of preferences on the component level can be aggregated to a cardinal ordering on the system level.

I. Reliability and the Arrow Impossibility Theorem

We will show in this chapter how the Arrow impossibility theorem can be proved using reliability theory arguments. Arrow [1951] considered the problem of aggregating the preference orderings of n individuals over a set of alternatives A to form a "social" preference ordering over the set A. Arrow formulated five reasonable conditions and two axioms that such an aggregation rule should satisfy. The result was the well known impossibility theorem which shows that no function exists that can satisfy all five conditions.

Consider a set of alternatives $A = \{a_1, \ldots, a_m\}$. We will denote by $a_1 P_i a_2$ the statement "individual i prefers a_1 to a_2." Similarly, $a_1 I_i a_2$ will mean that "individual i is indifferent between alternatives a_1, a_2." Finally, $a_1 R_i a_2$ is the negation of $a_2 P_i a_1$. In the following we will assume that each of the n individuals can order the alternatives presented to him in a complete ordering. That is

(i) Transitivity:

$a_1 R_i a_2$, $a_2 R_i a_3 \Rightarrow a_1 R_i a_3$, where $a_1, a_2, a_3 \in A$

(ii) Connectedness:

For any $a_1, a_2 \in A$ either $a_1 R_i a_2$, or $a_2 R_i a_1$, or both.

Once we assume the above, we can define a function

$$x_i(a_1,a_2) = \begin{cases} 1 & a_1 R_i a_2 \\ 0 & a_2 P_i a_1 \end{cases}$$

for each individual i.

We seek to find a social choice function. By that we mean a function $F_A(a_1,a_2)$ where $a_1, a_2 \in A$ and

$$F_A(a_1,a_2) = \begin{cases} 1 & \text{if society does not prefer } a_2 \text{ to } a_1 \\ & \text{(i.e., } a_1Ra_2\text{)} \\ 0 & \text{if society prefers } a_2 \text{ to } a_1 \\ & \text{(i.e., } a_2Pa_1\text{)} \end{cases}$$

The subscript A reminds us that F may depend on the whole set of alternatives A even though the comparison is between two elements of A.

The axioms and conditions imposed by Arrow on $F_A(\cdot,\cdot)$ are the following.

Axiom 1 (Connectedness):

For any $a_1,a_2 \in A$, $F_A(a_1,a_2)$ is either equal to 1 or 0 (in other words, for any two alternatives a_1,a_2 society either prefers a_1 to a_2 or a_2 to a_1 or is indifferent between them).

Axiom 2 (Transitivity):

If $F_A(a_1,a_2) = 1$, $F_A(a_2,a_3) = 1$ then $F_A(a_1,a_3) = 1$.

Condition 1:

(a) The number of elements in A is greater than or equal to three.

(b) The social choice function F is defined for all individual orderings.

(c) There are at least two individuals.

Condition 2 (Positive responsiveness):

If the social choice function is $F_A(a_1,a_2) = 1$ for a given set of individual preference orderings, it shall not decrease if the individual orderings are changed as follows:

(a) The individual paired comparisons between alternatives other

than a_1 remain the same and

(b) Each individual paired comparison between a_1 and any other alternative either remains unchanged or it is modified in favor of a_1 .

Condition 3 (Independence of irrelevant alternatives):

If an alternative is added or subtracted from the set of alternatives A the resulting social ordering must keep the alternatives in A in the same preference ordering.

Condition 4:

The social preference function must depend on the individual preference orderings only. And for any two alternatives a_1,a_2 there are individual preference orderings such that society prefers a_1 to a_2 .

Condition 5 (No dictator):

There is no individual with the property that whenever he prefers a_1 to a_2 (for any a_1,a_2) society prefers a_1 to a_2 regardless of the preferences of the other individuals.

A discussion of these conditions of which the weakest seems to be Condition 3, can be found in Luce and Raiffa [1957].

We will show now that

Lemma 1:

Conditions 2, 3, 4 imply that $F_A(a_1,a_2)$ can be written as a function $\phi(x_1(a_1,a_2), x_2(a_1,a_2), \ldots x_n(a_1,a_2))$ and further that $\phi(x_1 \ldots x_n)$ is a coherent structure.

The opposite, that when $F_A(a_1,a_2) = \phi(x_1 \ldots x_n)$ with ϕ coherent then conditions 2, 3, 4 are satisfied is immediate.

Proof:

Condition 3 implies that $F_A(a_1,a_2)$ does not depend on A . Thus we write simply $F(a_1,a_2)$.

Condition 4 requires that $F(a_1,a_2) = \phi(x_1(a_1,a_2) \ldots x_n(a_1,a_2))$.

Condition 2 requires that $\phi(\underline{x})$ is non-decreasing in each x_i .

Condition 4 again since $\phi(\underline{x})$ is non-decreasing requires that $\phi(\underline{0}) = 0$ and $\phi(\underline{1}) = 1$, where $\underline{x} \equiv x_1 \ldots x_n$, $\underline{0} \equiv \underbrace{0 \ldots 0}_{n}$, $\underline{1} \equiv \underbrace{1, \ldots, 1}_{n}$.||

Theorem 1 (Arrow's impossibility theorem):

Conditions 1, 2, 3, 4, 5 and Axioms I and II are inconsistent.

Discussion:

Lemma 1 has limited our search for a social choice function to coherent structures. Let us now introduce Axiom 1. Axiom 1 would be satisfied for a given coherent system if the individual orderings were such that for any two alternatives there is a path and a cut whose components (individuals) agree on the ordering of the two alternatives in question. Turning now to Axiom 2 (transitivity) we can see that the social choice function will satisfy it if the individual (component) orderings are such that for each triplet of alternatives a_1,a_2,a_3 there is a cut (not necessarily the same) which does not allow a cycle of the alternatives a_1,a_2,a_3 to pass. But the above describe relations between the individual preference structures and the structure function ϕ . We want, however, to find what

coherent systems if any satisfy Axioms 1 and 2 for *any* preference structure of the components (Condition 1).

Before we continue with the proof we will need some more notation. The *dual system* is defined as

$$\phi^D(\underline{x}) \equiv 1 - \phi(\underline{1} - \underline{x})$$

where

$$\underline{1} - \underline{x} \equiv 1 - x_1, 1 - x_2, \ldots, 1 - x_n$$

From reliability theory we know that:

(a) The dual of the dual is the original system.

(b) The minimal path (cut) sets of the system are the minimal cut (path) sets of the dual.

(c) If $\phi^D(\underline{x}) = \phi(\underline{x})$ for all $\underline{x}$ it follows that each minimal path set of $\phi(\underline{x})$ is also a minimal cut set of $\phi(\underline{x})$.

One such class of systems is the $\frac{n+1}{2}$-out-of-n system with n odd (the odd majority vote systems). But it is not the only one, for example, consider a 2 out of 3 system each component of which is itself a 2 out of 3 system. We can see that $\phi^D(\underline{x}) = \phi(\underline{x})$ but it is not a majority system. We now define the following classes of coherent systems.

S = class of coherent systems such that any two paths have at least one common component.

M = the class of coherent systems whose structure function $\phi(\underline{x})$ satisfies the identity

$$\phi^D(\underline{x}) = \phi(\underline{x}) \quad \text{for all } \underline{x} .$$

(We will often call these systems *self-dual*.)

P = class of coherent systems such that *all* paths have at least one common component.

We can immediately see that $P \subset S$. Also $M \subset S$ because for systems in M each path is also a cut; thus if two paths had no common component there would exist a cut (one of the two paths) which would not have a common component with a path (the other path) but this is impossible by definition of a cut.

Now we can return to the proof of the Theorem.

1) At first glance Axiom 1 requires that society is consistent, i.e., that society cannot both agree that "a_1 is preferred to a_2" and "a_1 is not preferred to a_2." This part of Axiom 1 limits our search for admissible systems to those in the class S since for any system in S any two paths have a common component which by assumption is consistent.

2) Axiom 1 further restricts us to the class M . This is so because we want that for any two alternatives a_1, a_2 society must either agree to the affirmative of "a_1 is preferred to a_2" or to the negation of this statement. If the affirmative induces the individual's response $\underline{x}$ the negation induces $\underline{1} - \underline{x}$ and we require for society that for any $\underline{x}$ either (a) or (b) must hold.

(a) $\phi(\underline{x}) = 1$ and $\phi(\underline{1} - \underline{x}) = 0$

(b) $\phi(\underline{x}) = 0$ and $\phi(\underline{1} - \underline{x}) = 1$.

(a) and (b) are equivalent to $1 - \phi(\underline{1} - \underline{x}) = \phi(\underline{x}) \Leftrightarrow \phi^D(\underline{x}) = \phi(\underline{x}) \Leftrightarrow \phi \in M$.

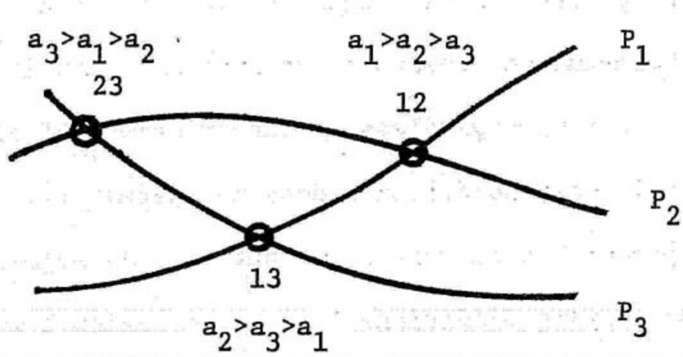
3) We argued in (1) above that the consistency part of Axiom 1 restricts us to the class S . Let us now find the systems in S that satisfy Axiom 2. We will show that it is the class P . That systems in P satisfy Axiom 2 is obvious once the common component observes transitivity as it is assumed.

We therefore focus our attention to showing that Axiom 2 and Condition 1 imply restriction to the class $\mathcal{P}$.

(i) If there is only one path the system belongs to $\mathcal{P}$ and satisfies Axiom 2 trivially.

(ii) If there are only two paths, since the system belongs in $\mathcal{S}$ they have a common component, thus again the system belongs to $\mathcal{P}$ and satisfies Axiom 2.

(iii) If there are three or more paths then for the moment pick any three of them P_1, P_2, P_3 . We know that any two of them have a common component. If they have more than one common component, choose their preference orderings to be identical. Thus we have only one common component between two paths. Let component (12) be common to paths P_1 and P_2 similarly (13) to P_1, P_3, (23) to P_2, P_3 . It is now easy to apply the well known example for non-transitivity: Let component's (12) preference ordering of the triplet a_1, a_2, a_3 be $a_1 P a_2 P a_3$. For component (13) $a_2 P a_3 P a_1$ and for (23) $a_3 P a_1 P a_2$. A picture will help:



By appropriate choice of the individual orderings in P_1 the path P_1 will agree to "a_2 is preferred to a_3" thus the system will have $F(a_2,a_3) = 1$. Similarly the system will agree to "a_3 is preferred to a_1" or $F(a_3,a_1) = 1$ because of P_3 . Finally because of P_2 , $F(a_1,a_2) = 1$. However $F(a_2,a_1) = 0$ since no path agrees that "a_2 is preferred or indifferent to a_1 ." Thus combining $F(a_2,a_3) = 1$, $F(a_3,a_1) = 1$ and $F(a_2,a_1) = 0$ and obtains violating Axiom 2. Thus, all three paths must have a common point, call it (123).

If there are more than three paths consider path P_4 and P_1P_2 . Now P_4 must pass through 12 but also P_3 must pass through 12. Repeating this argument we have that all paths must pass through 12 or that the system must belong to $\mathcal{P}$.

(iv) To satisfy therefore both Axioms 1 and 2 in view of Condition 1 when we are restricted to look in the coherent structures (Conditions 2, 3, 4) we must look at the intersection of the classes $\mathcal{M}$ and $\mathcal{P}$ $\mathcal{M} \cap \mathcal{P}$.

But now all systems in $\mathcal{P}$ have a one component cut (since a component say j belongs to all paths) while $\mathcal{M}$ requires that all cuts (paths) are also paths (cuts) of the system. Then j is both a cut and a path. Then j is the only relevant component in the system. That is, if $x_j = 1 \Rightarrow \phi = 1$ and if $x_j = 0 \Rightarrow \phi = 0$ regardless of the orderings of all other components. But this is exactly what Condition 5 does not permit.||

As a side product we can now show that the odd majority systems[1] are the only ones satisfying Conditions 1, 2, 3, 4, Axiom 1 and *symmetry* of the components. The proof is simple. We already know that the only systems that

[1] By odd majority systems we mean an $\frac{n+1}{2}$ out-of-n system with n odd.

satisfy Conditions 1, 2, 3, 4 and Axiom 1 are the self-dual systems. *Symmetry* requires that $\phi(x_1 \ldots x_n)$ is the same for any permutation of the arguments $x_1 \ldots x_n$. Now we proceed by induction. Certainly for a three component system $\phi^D = \phi$ implies that ϕ is a two out-of-three system. Now let an n (odd) component system $\phi_n(\underline{x})$ and suppose that $\phi_n^D = \phi_n$ along with symmetry implies that ϕ is a majority system. Let ϕ_{n+2} be a $n + 2$ component system, which satisfies

(1) $\phi^D_{n+2}(\underline{x}) = \phi_{n+2}(\underline{x})$ for all $\underline{x}$ and symmetry. It follows that for every i,j we can write

(2) $\phi_{n+2}(1_i;0_j;\underline{x}') = \phi^D_{n+2}(1_i;0_j;\underline{x}')$ for all $\underline{x}'$ where

$$1_i;0_j;\underline{x}' \equiv x_1, \ldots, x_{i-1},1,x_{i+1}, \ldots x_{j-1},0,x_{j+1}, \ldots; x_{n+2}$$

and

$$\underline{x}' \equiv x_1, \ldots, x_{i-1},x_{i+1}, \ldots, x_{j-1},x_{j+1}, \ldots, x_{n+2} .$$

Now consider $\phi_{n+2}(1_i;0_j;\underline{x}')$. It is an n component system once components i,j have a fixed value. Denote it $\phi_{n,1_i0_j}(\underline{x}')$. The dual of this system is defined as

$$\text{(3)} \qquad \phi^D_{n,1_i0_j}(\underline{x}') \equiv 1 - \phi_{n,1_i0_j}(\underline{1} - \underline{x}') .$$

But by definition the right hand side of (3) is

$$\text{(4)} \quad 1 - \phi_{n,1_i0_j}(\underline{1}-\underline{x}') = 1 - \phi_{n+2}(1_i;0_j;\underline{1}-\underline{x}') = \phi^D_{n+2}(0_i;1_j;\underline{x}')$$

but because of (2)

$$\text{(5)} \qquad = \phi_{n+2}(0_i;1_j;\underline{x}') = \phi_{n,0_i1_j}(\underline{x}')$$

or finally

(6) $$\phi^D_{n,1_i0_j}(\underline{x}') = \phi_{n,0_i1_j}(\underline{x}')$$

but because of symmetry

$$\phi_{n+2}(1_i0_j\underline{x}') = \phi_{n+2}(0_i1_j\underline{x}')$$

or

$$\phi_{n,0_i1_j}(\underline{x}') = \phi_{n,1_j0_i}(\underline{x}') .$$

Thus (6) becomes

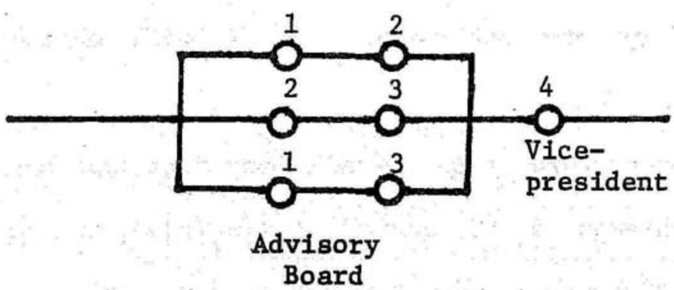
(7) $$\phi^D_{n,1_i0_j}(\underline{x}') = \phi_{n,1_i0_j}(\underline{x}') .$$

But because of the induction assumption (7) along with symmetry implies that $\phi_{n,1_i0_j}(\underline{x}')$ is an odd majority system. Now we have shown that the $n + 2$ system will behave as follows: For any $\underline{x}$ such that two out of the $n + 2$ components are one equal to zero and the other equal to one, the whole system will be one if an odd majority system of the rest of the components is one and zero otherwise. But this exactly says that the $n + 2$ component system is an odd majority system. The cases where all the components are equal to 1 or all 0 are trivial.

II. Coherent Decision Structures

2.1 General

Coherent structures are often used as decision structures in society, whether this is a government structure or the management of the corporation the problem is similar. In order for a proposal to be accepted by a decision structure it has to pass through several channels. Along the way components study it and either accept it or reject it. If all components along a path accept it, then the proposal passes. For example, consider an advisory board of three people and a vice-president of a corporation. A proposal passes if it passes through the advisory board with a simple majority and the vice president agrees. The system can be represented by:



and the structure function will be:

$$\phi(\underline{x}) = (x_1 + x_2 + x_3 - 2x_1x_2x_3)x_4 .$$

Consider now n individuals and a set of alternatives $A = \{a_1, \ldots, a_n\}$. We will assume that each individual for any alternative $a_1 \varepsilon A$ either considers "a_1 is a best alternative in A ," or "a_1 is not a best alternative in A ." By "best" we mean that a_1 is preferred or indifferent to all other alternatives in A . This implies that with each individual i and a set of alternatives A we can associate a function

$$x_i(a_j|A) \equiv \begin{cases} 1 & \text{if "}a_j \text{ is a best alternative in A"} \\ 0 & \text{if "}a_j \text{ is not a best alternative in A"} \end{cases}$$

$a_j \varepsilon A$.

Note that the $x_i(\cdot)$ depends in general on the set of alternatives A . By this we want to imply that the individuals may not respect the irrelevant alternatives condition. For example, individual i may consider a_1 as the best among $\{a_1,a_2,a_3\}$ but also may consider a_2 as the best when his set of alternatives is limintd to $\{a_1,a_2\}$.

We will denote

$$\hat{A}_i \equiv \{a \varepsilon A \mid x_i(a|A) = 1\}$$

and call it the *optimal set* for individual i . We will assume that $\hat{A}_i$ is non-empty for any set A .

Given a coherent structure $\phi(\underline{x})$ we will say that the statement "a is a best in A" *passes* through ϕ if $\phi(\underline{x}(a)) \equiv \phi(x_1(a|A), x_2(a|A), \ldots, x_n(a|A)) = 1$ and "a is a best in A" does *not pass* through ϕ if $\phi(\underline{x}(a)) = 0$.

Consider however the negation of "a is a best alternative in A" . Each component in the system when faced with the statement "a is not a best in A" will take the value $1 - x_i(a|A)$ which will be one if he agrees and zero if he doesn't. At the system level we will then say that "a is not a best in A" passes (does not pass) through $\phi(\underline{x})$ if

$$\phi(\underline{1} - \underline{x}(a|A)) = 1(0) .$$

There are many coherent structures which will pass both the affirmative and the negation of a statement (consider for example the parallel system)

as well as many others that will not pass either the affirmative or the negation of a statement (example: series system).

In general therefore for a set of alternatives A a system can for some alternatives pass the affirmative and the negation or the affirmative but not the negation, etc. We will distinguish between alternatives by using the *indicator* systems $\rho_+(\underline{x})$, $\rho_-(\underline{x})$, $\rho_B(\underline{x})$, $\rho_C(\underline{x})$ to be defined in the following.

Definition 1:

The *system* $\phi(\underline{x})$ *strongly considers* a as a best alternative in A iff

$$\phi(\underline{x}(a|A)) = 1 \ \ \underline{\text{and}} \ \ \phi(\underline{1} - \underline{x}(a|A)) = 0 \ .$$

That is, if "a is best" passes and "a is not best" does not pass.

Definition 1':

$$\rho_+(\underline{x}) \equiv \phi(\underline{x})\phi^D(\underline{x}) \ .$$

It is then clear that $\phi(\underline{x})$ *strongly consider* a as best iff $\rho_+(\underline{x}(a|A)) = 1$.

Definition 2:

We say that the system $\phi(\underline{x})$ *strongly considers* a as not a best alternative in A iff

$$\phi(\underline{x}(a|A)) = 0 \ \ \underline{\text{and}} \ \ \phi(\underline{1} - \underline{x}(a|A)) = 1 \ .$$

That is if "a is best" does not pass and "a is not best" passes.

Definition 2':

$$\rho_-(\underline{x}) \equiv \phi(\underline{1} - \underline{x})\phi^D(\underline{1} - \underline{x}) \ .$$

It is now clear that $\phi(\underline{x})$ *strongly considers* a not a best alternative, iff $\rho_-(\underline{x}(a|A)) = 1$.

Definition 3:

The system $\phi(\underline{x})$ is *blocked* for a iff

$$\phi(\underline{x}(a|A)) = 0 \quad \text{and} \quad \phi(\underline{1} - \underline{x}(a|A)) = 0 .$$

That is, when both statements "a is best," "a is not best" do not pass.

Definition 3':

$$\rho_B(\underline{x}) \equiv \phi(\underline{1} - \underline{x})\phi^D(\underline{x})$$

then the system is *blocked* for a iff $\rho_B(\underline{x}(a|A)) = 1$.

Definition 4:

The system is *contradictory* for a iff

$$\phi(\underline{x}(a|A)) = 1 \quad \text{and} \quad \phi(\underline{1} - \underline{x}(a|A)) = 1 .$$

That is, both statements "a is best" and "a is not best" pass.

Definition 4':

$$\rho_C(\underline{x}) \equiv \phi(\underline{x})\phi(1 - \underline{x})$$

Then the system $\phi(\underline{x})$ is contradictory with respect to a iff $\rho_C(\underline{x}(a|A)) = 1$.

The indicator systems divide the set of alternatives A into four mutually exclusive and conclusive sets which we will call *characteristic sets* of system ϕ over the set A .

$$A_+(A) \equiv \{a \in A | \rho_+(\underline{x}(a|A)) = 1\}$$
$$A_-(A) \equiv \{a \in A | \rho_-(\underline{x}(a|A)) = 1\}$$
$$A_B(A) \equiv \{a \in A | \rho_B(\underline{x}(a|A)) = 1\}$$
$$A_C(A) \equiv \{a \in A | \rho_C(\underline{x}(a|A)) = 1\}$$

We will also define

$$A \equiv \{a \in A | \phi(\underline{x}(a)) = 1\} .$$

Certainly $A = A_+ \cup A_C$ since $\phi(\underline{x}) = \rho_+(\underline{x}) + \rho_C(\underline{x})$.

2.2 Some Properties of the Indicator Systems

Property 1:

$\rho_+(\underline{x})$ is coherent if $\phi(\underline{x})$ is coherent.

Proof:

$\phi(\underline{x})$ coherent implies $\phi^D(\underline{x})$ coherent and the series system of two coherent systems is also coherent.

Property 2:

$\rho_-(\underline{x})$ is "anti-coherent" if $\phi(\underline{x})$ is coherent. (By anti-coherent we mean that $\rho_-(\underline{x})$ is non-increasing in $\underline{x}$ and $\rho_-(\underline{1}) = 0$, $\rho_-(\underline{0}) = 1$. In other words $1 - \rho_-(\underline{x})$ is coherent).

Proof:

Similar.

Property 3: Computational properties. We list them without proof.

3.1) $\rho_+^D(\underline{x}) + \rho_+(\underline{x}) = \phi^D(\underline{x}) + \phi(\underline{x})$

3.2) $\rho_+^D(\underline{x}) = 1 - \rho_-(\underline{x})$

3.3) $\rho_+(\underline{1} - \underline{x}) = \rho_-(\underline{x})$

3.4) $\rho_-^D(\underline{x}) = 1 - \rho_+(\underline{x})$

3.5) $\rho_-^D(\underline{x}) + \rho_-(\underline{x}) = \phi(\underline{1} - \underline{x}) + \phi^D(\underline{1} - \underline{x})$

3.6) $\rho_B^D(\underline{x}) = 1 - \rho_B(\underline{x})$

3.7) $\rho_B(\underline{1} - \underline{x}) = \rho_B(\underline{x})$

3.8) $\rho_B^D(\underline{x}) + \rho_C(\underline{x}) = \phi(\underline{1} - \underline{x}) + \phi(\underline{x})$

3.9) $\rho_C^D(\underline{x}) + \rho_B(\underline{x}) = \phi^D(\underline{1} - \underline{x}) + \phi^D(\underline{x})$

3.10) $\rho_C^D(\underline{x}) = 1 - \rho_C(\underline{x})$

3.11) $\rho_C(\underline{1} - \underline{x}) = \rho_C(\underline{x})$

2.3 The Classes S, $\bar{S}$ and M

Let C be the class of coherent systems. Define S as the class of coherent systems such that

$$S \equiv \{\phi(\underline{x}) \in C \mid \rho_C(\underline{x}) = 0, \forall \underline{x}\}$$

In other words systems in S are never inconsistent. Also define

$$\bar{S} \equiv \{\phi(\underline{x}) \in C \mid \rho_B(\underline{x}) = 0 \ , \forall \underline{x}\} \ .$$

Then $\bar{S}$ are the systems that are never blocked.

The indicator systems take special forms for systems in S, $\bar{S}$ and M. These properties are summarized below without proofs.

1) $\phi \in S \iff \rho_C = 0$

$\rho_+(\underline{x}) = \phi(\underline{x})$

$\rho_B(\underline{x}) = \phi^D(\underline{x}) - \phi(\underline{x})$

$\rho_-(\underline{x}) = \phi(\underline{1} - \underline{x})$

2) $\phi \; \varepsilon \; \bar{S} \iff \rho_B = 0$

$\rho_+(\underline{x}) = \phi^D(\underline{x})$

$\rho_C(\underline{x}) = \phi(\underline{x}) - \phi^D(\underline{x})$

$\rho_-(\underline{x}) = \phi^D(\underline{1} - \underline{x})$

3) $\phi \; \varepsilon \; \mu \iff \phi = \phi^D \iff \rho_B = 0, \; \rho_C = 0$

$\rho_+(\underline{x}) = \phi(\underline{x})$

$\rho_-(\underline{x}) = \phi(\underline{1} - \underline{x})$

Lemma 2.1:

$\phi \; \varepsilon \; S$ if and only if $\phi^D \; \varepsilon \; \bar{S}$.

Proof:

$$\phi \; \varepsilon \; S \Rightarrow \rho_C = 0 \Rightarrow \phi(x)\phi(\underline{1} - x) = 0 \; ; \forall \underline{x}$$

Now look at ρ_B of the dual system:

$$\rho_B = (\phi^D(\underline{x}))^D(\phi^D(\underline{1} - \underline{x}))^D = \phi(\underline{x})\phi(\underline{1} - \underline{x}) = 0$$

Thus $\phi^D(\underline{x}) \; \varepsilon \; \bar{S}$. The opposite is proved similarly.

Lemma 2.2:

The definition of S is equivalent to the definition of S in Chapter I (i.e., S are the systems for which any two paths have a common component.)

Proof:

(i) Take $\phi(\underline{x})$ with $\rho_C(\underline{x}) = 0$, $\forall \underline{x}$. Then suppose $\phi(\underline{x})$ has two paths with no common component. Then by choice of $\underline{x}$ I can make all elements in the one path equal to 1 and all elements in the other path equal to zero.

Then $\phi(\underline{x}) = 1$ and $\phi(\underline{1} - \underline{x}) = 1 \Rightarrow \rho_C(\underline{x}) = 1$ for some $\underline{x}$. Contradiction.

(ii) Take $\phi(\underline{x})$ such that any two paths have a common component. Then suppose that for some $\underline{x}$, $\rho_C(\underline{x}) = 1$. This implies $\phi(\underline{x}) = 1$ and $\phi(\underline{1} - \underline{x}) = 1$ which implies that $\phi(\underline{x})$ has a path with all components equal to 1 and a path with all components equal to zero for that $\underline{x}$. Thus there are two paths with no common component. Contradiction.

Lemma 2.3:

$\bar{S}$ consists of coherent systems such that any two cuts have a common component and only those.

Proof:

Since $\phi \in \bar{S} \Leftrightarrow \phi^D \in S$ and since that paths of the dual are the cuts of the system and vice versa, it follows by Lemma 2, that ϕ must be such that any two cuts have a common component.

Lemma 2.4:

The only systems that are never blocked or inconsistent are the systems in M (i.e., the systems such that $\phi(\underline{x}) = \phi^D(\underline{x})$, $\forall \underline{x}$).

Proof:

(i) If $\phi(\underline{x}) = \phi^D(\underline{x})$ then

$$\rho_C(\underline{x}) = \phi(\underline{x})\phi(\underline{1} - \underline{x}) = \phi(\underline{x})(\underline{1} - \phi^D(\underline{x})) = \phi(\underline{x}) - \phi(\underline{x})\phi^D(\underline{x}) =$$
$$= \phi(\underline{x}) - \phi(\underline{x}) = 0 .$$
$$\rho_B(\underline{x}) = \phi^D(\underline{x})\phi^D(\underline{1} - \underline{x}) = \phi(\underline{x})\phi(\underline{1} - \underline{x}) = \rho_C(\underline{x}) = 0 .$$

(ii) If $\rho_C(\underline{x}) = 0$ and $\rho_B(\underline{x}) = 0 \ \forall \underline{x}$

Then

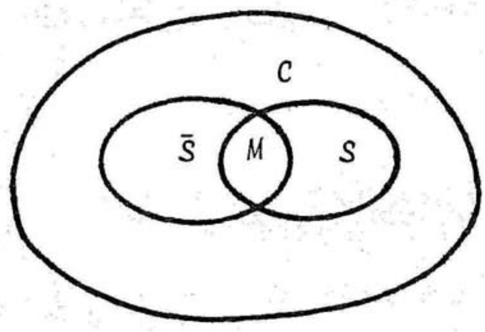
$$\rho_C(\underline{x}) = \phi(\underline{x})\phi(\underline{1} - \underline{x}) = 0$$

$$\rho_B(\underline{x}) = \phi^D(\underline{x})\phi^D(\underline{1} - \underline{x}) = (1 - \phi(\underline{1} - \underline{x}))(1 - \phi(\underline{x})) = 1 + \phi(\underline{x})\phi(\underline{1} - \underline{x}) - \phi(\underline{x}) - \phi(\underline{1} - \underline{x}) = 0 \ .$$

Substituting the first in the second equation we obtain:

$$1 - \phi(\underline{x}) - \phi(\underline{1} - \underline{x}) = 0 \Rightarrow \phi(\underline{x}) = \phi^D(\underline{x}) \ .||$$

Pictorially we have:



$M = \bar{S} \cap S$

It is also immediate that for systems in M any path is also a cut and vice versa.

Some examples of systems in M are the odd majority systems ($\frac{n+1}{2}$ out-of-n , with n odd) and odd majority systems whose components are odd majority systems. For example, a 2 out of 3 system of 2 out of 3 systems.

It is now obvious that the series combination of two S systems is also in S , while the parallel combination of two systems in $\bar{S}$ is also in $\bar{S}$. Whether a parallel combination of two S-systems is also in S will depend on the structure of the two systems. The following propositions deal with this question.

Proposition 2.1:

Let $\phi_1(\underline{x})$, $\phi_2(\underline{y}) \in S$ where $\underline{x}$ and $\underline{y}$ have at least one common component. Then their parallel combination will still belong to S if and only if $\phi_1(\underline{x})\phi_2(\underline{1} - \underline{y}) = 0$ for all $\underline{x}$, $\underline{y}$.

Proof:

The new system will have a structure function

$$\phi(\underline{x},\underline{y}) = \phi_1(\underline{x}) + \phi_2(\underline{y}) - \phi_1(\underline{x})\phi_2(\underline{y})$$

We calculate ρ_C for this system,

$$\rho_C = \phi(\underline{x},\underline{y})\phi(\underline{1} - \underline{x}, \underline{1} - \underline{y})$$

Substituting for $\phi(\underline{x},\underline{y})$ and noting that $\phi_1(\underline{x})\phi_1(\underline{1} - \underline{x}) = 0$, $\phi_2(\underline{y})\phi_2(\underline{1} - \underline{y}) = 0$ since $\phi_1,\phi_2 \in S$ we obtain

$$\rho_C = \phi_2(\underline{y})\phi_1(\underline{1} - \underline{x}) + \phi_2(\underline{1} - \underline{y})\phi_1(\underline{x})$$

But then to require that $\rho_C = 0$ (which is equivalent to the parallel combination belonging to S) is equivalent to asking that:

$$\phi_2(\underline{y})\phi_1(\underline{1} - \underline{x}) = 0 \quad \text{and} \quad \phi_2(\underline{1} - \underline{y})\phi_1(\underline{x}) = 0 \quad \forall \underline{x},\underline{y} \ .$$

But this is equivalent to one only condition:

$$\phi_1(\underline{x})\phi_2(\underline{1} - \underline{y}) = 0 \quad \forall \underline{x},\underline{y} \ . \ ||$$

Remark:

The requirement that $\underline{x}$ and $\underline{y}$ have some common components is needed because otherwise the condition we found would be satisfied only for trivial systems like $\phi_1(\underline{x}) = 1$, $\phi_2(\underline{y}) = 0$ $\forall \underline{x},\underline{y}$.

Proposition 2.2:

Let $\phi_1(\underline{x})$, $\phi_2(\underline{y}) \in \bar{S}$ where $\underline{x}$ and $\underline{y}$ have at least one common component. Then their series combination will still belong to $\bar{S}$ if and only if $\phi_1^D(\underline{x})\phi_2^D(1 - y) = 0$ for all $\underline{x}, \underline{y}$.

Proof:

Similar to the above only now:

$$\phi(\underline{x},\underline{y}) = \phi_1(\underline{x})\phi_2(\underline{y}) \quad \text{and} \quad \rho_B = \phi^D(\underline{x},\underline{y})\phi^D(\underline{1} - \underline{x},\underline{1} - \underline{y}) \ .||$$

Let us look a little closer at the conditions of Proposition 2.1 and 2.2. For Proposition 2.1 the condition is $\phi_1(\underline{x})\phi_2(\underline{1} - y) = 0 \iff \phi_1(\underline{x})(\underline{1} - \phi_2^D(\underline{y})) = 0 \iff \phi_2^D(\underline{y})\phi_1(\underline{x}) = \phi_1(\underline{x}) \iff \phi_2^D(\underline{x}) \geq \phi_1(\underline{x}) \iff$ by symmetry $\phi_1^D(\underline{x}) \geq \phi_2(\underline{y})$.

Some systems that satisfy this are:

1) $\phi_2^D(\underline{y})$ is a k out-of-n system.

$\phi_1(\underline{x})$ is an ℓ out-of-n system where $\underline{x} = \underline{y}$ and $\ell \geq k$.

2) Let $\underline{z}$ be common in both $\underline{y}$ and $\underline{x}$. Then $\phi_2^D(\underline{y})$ is a k out-of-n system over $\underline{z}$ in parallel with any other system over $\underline{y}$ and $\phi_1(\underline{x})$ is an ℓ out-of-n system over $\underline{z}$ in series with any other system over $\underline{x}$ *and* $\ell \geq k$.

A real life situation of the latter example is the governmental system (President-Congress) in the U. S. Here ϕ_1 is the House (1/2 majority) in series with the Senate (1/2 majority) in series with the president. While ϕ_2 is the House (2/3 majority) in series with the Senate (2/3 majority) the dual of ϕ_2 is the dual of the House (i.e., House with 1/3 majority) in parallel with the Senate (1/3 majority) and indeed, 1/2 > 1/3 majority. We can now

safely conclude that this government belongs to the S systems (possibly blocked but never inconsistent). At least this is something.

Similarly the condition of Proposition 2.2 is equivalent to $\phi_2(\underline{y}) \geq \phi_1^D(\underline{x})$ or $\phi_1(\underline{x}) \geq \phi_2^D(\underline{y})$ and similar examples hold.

2.4 Properties of the Characteristic Sets

Theorem 2.1:

If for a set of alternatives A, each component of a coherent system $\phi(\underline{x})$ is allowed to consider one and only one alternative as the best (i.e., the optimal set $\hat{A}_i$ of each individual $i = 1, \dots n$ contains one and only one element) then,

(a) $A_+(A)$ can contain at most one element.

(b) If $A_+(A) \neq \emptyset$ then $A_B(A) = \emptyset$, and $A_C(A) = \emptyset$

(c) If $A_B(A) \neq \emptyset$ then $A_C(A) = \emptyset$

Proof:

(a) First it is clear that $A_+(A)$ can be empty. Consider for example a series system where not all components agree on which is the best alternative in A.

Now if $a \in A$ is such that $\rho_+(\underline{x}(a)) = 1$ this means that $\phi(\underline{x}(a)) = 1$ and $\phi^D(\underline{x}(a)) = 1$ or that there is a path and a cut in $\phi(\underline{x})$ that considers a as the best in A. Since the components are not allowed to consider more than one alternative as the best it follows that a is the only element in $A_+(A)$.

(b) If $A_+(A) \neq \emptyset$ let a be the element in $A_+(A)$. Now suppose that for $b \neq a$ $\rho_B(\underline{x}(b)) = 1$ then $\phi^D(\underline{1} - \underline{x}(b)) = 1$ and

$\phi^D(\underline{x}(b)) = 1$. The latter implies that there is a path in the dual that considers b the best or a cut in the primal $(\phi(\underline{x}))$ that considers b the best. But since $\rho_+(x(a)) = 1$ there is a path and a cut in $\phi(\underline{x})$ that considers a the best. Contradiction since the common element of path that prefers a and the cut that prefers b must prefer both.

To prove that $A_C(A) = \emptyset$ we again observe that $b \in A_C(A)$ implies that there is a path in $\phi(\underline{x})$ that considers b the best which contradicts the fact that a is the best for both a path and a cut in $\phi(\underline{x})$.

(c) If $A_B(A) \neq \emptyset$ then for $a \in A_B(A)$ $\rho_B(\underline{x}(a)) = 1 \Rightarrow$ there is a cut but not a path in $\phi(\underline{x})$ that considers a the best in A . Now if $A_C(A) \neq \emptyset$ there is $b \in A_C(A)$ or $\rho_C(\underline{x}(b)) = 1$ or there is a path but not a cut in $\phi(\underline{x})$ which regards b the best in A . But the above implies that the common component of the path and the cut must prefer both a and b in A ; contradiction. ||

<u>Corollary 2.1</u>:

If $|A| > 1$ and $A_-(A) = \emptyset$ then $A_+(A) = \emptyset$.

<u>Proof</u>:

Since $A_-(A) = \phi$, $A = A_+(A) \cup A_B(A) \cup A_C(A)$ but by Theorem 1 (b) if $A_+(A) \neq \emptyset \Rightarrow A_B(A) \cup A_C(A) = \emptyset$. This implies that $A = A_+(A)$ but then $A_+(A)$ must contain more than one element. Contradiction. Thus $A_+(A) = \emptyset$.

The above can be summarized in the following table:

	A_+	A_B	A_C	A_-
A_+	Yes			Yes
A_B		Yes		Yes
A_C			Yes	Yes
A_-	Yes	Yes	Yes	Yes

Figure 1

Summary of Theorem 1
(A "Yes" indicates that the sets of corresponding to the row and column can be both non-empty. All other combinations are impossible.)

Corollary 2.2:

For any given system $\phi(\underline{x})$ and set of alternatives A if each component considers one and only one alternative in A as the best then one and only one of the following statements holds:

(a) $A_+(A) \cup A_-(A) = A$

(b) $A_B(A) \cup A_-(A) = A$

(c) $A_C(A) \cup A_-(A) = A$

(d) $A_-(A) = A$

Proof:

Obvious from Theorem 2.1.

III. The Introduction of Probabilities

Once we formed coherent decision structures we might as well introduce probabilities. Let's assume that each individual (component) has a probability of agreeing that "a is a best alternative in A" for each $a \in A$. Mathematically we have,

$$v_i(a|A) \equiv \Pr[x_i(a|A) = 1] = Ex_i(a|A) .$$

If we assume that each individual can at any given time prefer one and only one alternative from the set A then

$$\sum_{a \in A} v_i(a|A) = 1 ; i = 1, \ldots, n .$$

We can now calculate the probabilities

$$P_+(a|A) \equiv \Pr[\rho_+(\underline{x}(a|A) = 1] = \Pr[a \in A_+(A)]$$

$$P_B(a|A) \equiv \Pr[\rho_B(\underline{x}(a|A)) = 1] = \Pr[a \in A_B(A)]$$

$$P_C(a|A) = \Pr[\rho_C(\underline{x}(a|A)) = 1] = \Pr[a \in A_C(A)]$$

$$P_-(a|A) \equiv \Pr[\rho_-(\underline{x}(a|A)) = 1] = \Pr[a \in A_-(A)]$$

Certainly $P_+(a|A) + P_-(a|A) + P_B(a|A) + P_C(a|A) = 1$

If we assume that $x_i(a|A)$, $x_j(a|A)$ are independent for any i,j then the above probabilities are functions of $v_1(a|A) \ldots v_n(a|A)$ only and they are linear in each $v_i(a|A)$; $i=1,\ldots,n$. Using standard reliability theory arguments we can show that

(a) $P_+(a|A)$ is non-decreasing in each $v_i(a|A)$

(b) $P_-(a|A)$ is non-increasing in each $v_i(a|A)$

(c) $P_+(a|A) + P_C(a|A)$ is non-decreasing in each $v_i(a|A)$ since it equals to $\Pr[\phi(\underline{x}(a|A)) = 1]$.

(d) $P_-(a|A) + P_B(a|A)$ is non-increasing in each $v_i(a|A)$ (because of (c)).

Further, the above functions are *strictly* increasing or decreasing if $0 < v_i(a|A) < 1$ for all i and there are no irrelevant[1)] components.

The relevance of the quantities P_+, P_-, P_B, P_C is obvious. They tell us the probabilities of a system being passing (P_+) blocked (P_B) inconsistent (P_C) or not passing (P_-) as a function of the alternative in question.

We can give however, another interesting interpretation. The individual probabilities $v_i(a|A)$ imply a complete ordering of the alternatives for each individual. Likewise, each of the probability functions $P_+(a|A)$, $P_-(a|A)$, $P_B(a|A)$, $P_C(a|A)$ orders the alternatives in the sense of strongly passing, strongly not passing, being blocked, being inconsistent respectively. It is easy to check that $P_+(a|A)$, $1 - P_-(a|A)$, $P_+(a|A) + P_C(a|A)$, $1 - P_-(a|A) - P_B(a|A)$ each satisfies Axioms 1 and 2 and Conditions 1, 2, 4, 5 of Arrow. Condition 3 (irrelevant alternatives) is not satisfied but then again it does not make sense to require it once we assumed that it does not hold on the individual level.

[The assumption that the condition of irrelevant alternatives might not hold on the individual level seems awkward at first. It is not. Consider for example, the three alternatives:

a_1: a raincoat with removable lining

a_2: the same raincoat as above without lining

a_3: a hat.

The question is asked, what is the probability "a_1 is the best in $\{a_1,a_2,a_3\}$" and the same question is asked for a_2 and a_3. Since a_1 dominates a_2 it is very likely that the probabilities will order the

1) A component i is *irrelevant* if $\phi(1_i\underline{x}) = \phi(0_i\underline{x})$ $\forall \underline{x}$ where $1_i\,\underline{x} \equiv x_1, x_2, \ldots 1_i, x_{i+1} \ldots, x_n$ and $0_i\underline{x} \equiv x_1, x_2 \ldots 0_i, x_{i+1} \ldots, x_n$.

alternatives as $P(a_1) > P(a_3) > P(a_2)$ where $P(a_i) \equiv Pr(a_i$ is the best in $\{a_1,a_2,a_3\})$; $i = 1,2,3 \ldots$]

In this sense any of the functions $P_+(a|A)$, $1 - P_-(a|A)$, $P_+(a|A) + P_C(a|A)$ can be considered as implying a cardinal ordering on the set of alternatives A. Still the question remains, which one is to be chosen. Concentrate for the moment on $P_+(a|A)$ and $P_+(a|A) + P_C(a|A)$. The first gives us the probability that a will pass as a best and not pass as not a best while the second gives us the probability the a will pass as a best (regardless if the negation of a also passes). This is a matter of definition of the system and what the system will pass as best. An example for the second kind of system is the following:

[A group of people follow a two stage procedure to elect an officer in their governing body. Let there be n people and $m < n$ candidates. In the first stage any candidate that can obtain $[\frac{n}{m}]$ of the votes is elected. In the second stage one of the candidates of the first stage is selected by some other procedure, say by chance[1]. Focus on the first stage. It is an $[\frac{n}{m}]$ out-of-n system and by definition we care only for the cases when $\phi(\underline{x}) = 1$ or for the $Pr[\phi(\underline{x}(a|A)) = 1] = P_+(a|A) + P_C(a|A)$. Thus the choice between $P_+(a|A)$ or $P_+(a|A) + P_C(a|A)$ depends on the definition of the system. The same argument holds if the choice were between $1 - P_-(a|A)$ and $P_+(a|A) + P_C(a|A)$.]

The real problem is then in the strict systems. That is, the systems that elect an alternative a only if $\rho_+(\underline{x}(a|A)) = 1$. In this realm it is not clear whether a decision analyst should try to maximize $P_+(a|A)$ over $a \in A$ or minimize $P_-(a|A)$ over $a \in A$. There is one case where the two

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[1] By the way, chance has many merits that modern societies don't realize. It was, however, effectively used for elections in ancient Athens.

approaches are equivalent: If the system belongs to M, then we know that $P_+(a|A) + P_-(a|A) = 1$ for all $a \in A$ for any A.

IV. Conclusions

The first chapter proved once more the Arrow Impossibility Theorem. The purpose was to show the relevance of coherent structures and reliability theory to problems of social choice. Continuing in this direction and to avoid the deadlock of the Impossibility Theorem we kept the concept of coherent structures but abolished the condition on irrelevant alternatives right at the individual level. It is important to note that we did not approach the problem by postulating conditions and then trying to find functions that aggregate individual preferences to social preferences because then again the question exists on how these functions are going to take the form of real life institutions. Our approach started in the opposite direction. We realized that many decision structures in real life have the form of coherent systems. We don't discuss the morality of such forms or why and how the positions on the structure are occupied by one individual and not another. Once the structure is given, however, we know that all the structure cares about is whether the components (individuals) agree or disagree with an issue that is fed into the system and not whether one component likes or dislikes the issue a lot or a little. With this attitude in mind we studied classes of systems as to whether they will be able to answer consistently or be blocked, and we showed that only the self-dual systems are never blocked or inconsistent.

With the introduction of probabilities on the individual level we were able to aggregate to the system level (through standard reliability procedures) and find what the probability of each alternative was for passing, being rejected, being blocked, etc. Seen from another point of view, the probabilities on the individual level induce a cardinal ordering

of alternative which is aggregated to a cardinal ordering on the systems level by P_+ or P_- or $P_+ + P_C$. In the case of self-dual systems of the above aggregation forms are equivalent.

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